

# A REMARK ON A PAPER OF P. B. DJAKOV AND M. S. RAMANUJAN

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**ABSTRACT.** Let  $\ell$  be a Banach sequence space with a monotone norm in which the canonical system  $(e_n)$  is an unconditional basis. We show that if there exists a continuous linear unbounded operator between  $\ell$ -Köthe spaces, then there exists a continuous unbounded quasi-diagonal operator between them. Using this result, we study in terms of corresponding Köthe matrices when every continuous linear operator between  $\ell$ -Köthe spaces is bounded. As an application, we observe that the existence of an unbounded operator between  $\ell$ -Köthe spaces, under a splitting condition, causes the existence of a common basic subspace.

## 1. INTRODUCTION

Following [2], we denote by  $\ell$  a Banach sequence space in which the canonical system  $(e_n)$  is an unconditional basis. The norm  $\|\cdot\|$  is called monotone if  $\|x\| \leq \|y\|$  whenever  $|x_n| \leq |y_n|$ ,  $x = (x_n)$ ,  $y = (y_n) \in \ell$ ,  $n \in \mathbb{N}$ . Let  $\Lambda$  be the class of such spaces with monotone norm. In particular,  $l_p \in \Lambda$  and  $c_0 \in \Lambda$ . It is known that every Banach space with an unconditional basis  $(e_n)$  has a monotone norm which is equivalent to its original norm. Indeed, it is enough to put

$$\|x\| = \sup_{|\beta_n| \leq 1} \left| \sum_n e_n'(x) \beta_n e_n \right|$$

where  $|\cdot|$  denotes the original norm,  $(e_n')$  denote the sequence of coefficient functionals.

Let  $\ell \in \Lambda$  and  $\|\cdot\|$  be a monotone norm in  $\ell$ . If  $A = (a_n^k)$  is a Köthe matrix, the  $\ell$ -Köthe space  $\lambda^\ell(A)$  is the space of all sequences of scalars  $(x_n)$  such that  $(x_n a_n^k) \in \ell$  with the topology generated by the

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2010 *Mathematics Subject Classification.* 46A45.

*Key words and phrases.* bounded operators, unbounded operators,  $\ell$ -Köthe spaces.

This research was partially supported by Turkish Scientific and Technological Research Council.

seminorms

$$\|(x_n)\|_k = \|(x_n a_n^k)\|$$

For any linear operator  $T : X \longrightarrow Y$  between Fréchet spaces we consider the following operator seminorms

$$\|T\|_{p,q} = \sup \left\{ \|Tx\|_p : \|x\|_q \leq 1 \right\}, \quad p, q \in \mathbb{N}$$

which may take the value  $+\infty$ . In particular, for any one dimensional operator  $T = u \otimes x$ , we have

$$\|T\|_{p,q} = \|u\|_q^* \|x\|_p$$

The operator  $T$  is continuous if and only if for all  $k$  there is  $N(k)$  such that

$$\|T\|_{k,N(k)} < \infty,$$

$T$  is bounded if and only if there is  $N \in \mathbb{N}$  such that for all  $r \in \mathbb{N}$ ,

$$\|T\|_{r,N} < \infty.$$

We write  $(X, Y) \in \mathcal{B}$  if every continuous linear operator on  $X$  to  $Y$  is bounded. Zahariuta [7] obtained that if the matrices  $A$  and  $B$  satisfy the conditions  $d_2$  and  $d_1$ , respectively, then  $(\lambda^{l_1}(A), \lambda^{l_1}(B)) \in \mathcal{B}$ . This phenomenon was studied extensively by Vogt [6] not only for Köthe spaces but also for the general case of Fréchet spaces. In case of  $\ell$ -Köthe spaces, there is no characterization of pairs  $(X, Y)$  with the property  $\mathcal{B}$ .

For Fréchet spaces  $X$  and  $Y$ , in [6], Vogt proved that  $(X, Y) \in \mathcal{B}$  if and only if for every sequence  $N(k)$ ,  $\exists N \in \mathbb{N}$  such that  $\forall r \in \mathbb{N}$  we have  $k_0 \in \mathbb{N}$  and  $C > 0$  with

$$(1.1) \quad \|T\|_{r,N} \leq C \max_{1 \leq k \leq k_0} \|T\|_{k,N(k)}$$

for all  $T \in \mathcal{L}(X, Y)$ .

An operator  $T : \lambda^\ell(A) \longrightarrow \lambda^\ell(B)$  is called quasi-diagonal if there exists  $k : \mathbb{N} \longrightarrow \mathbb{N}$  and constants  $m_n$  such that

$$Te_n = m_n \tilde{e}_{k(n)}, \quad n \in \mathbb{N}$$

Following [4], a pair of Köthe spaces  $(\lambda^\ell(B), \lambda^\ell(A))$  satisfies the condition  $\mathcal{S}$  if,

$$(1.2) \quad \forall p \quad \exists q, k \quad \forall s, l \quad \exists r, C : \frac{b_m^s}{a_n^k} \leq C \max \left\{ \frac{b_m^q}{a_n^p}, \frac{b_m^r}{a_n^l} \right\}$$

In [3] it was proved that the existence of an unbounded continuous linear operator from nuclear  $l_1$ -Köthe space to another implies the existence of a continuous unbounded quasi-diagonal operator. Also, if the both Köthe spaces are nuclear, in [5], Nurlu and Terzioğlu proved that

the existence of an unbounded continuous linear operator on  $\lambda^1(A)$  to  $\lambda^1(B)$  implies, under some conditions, the existence of a common basic subspaces of  $\lambda^1(A)$  and  $\lambda^1(B)$ . Djakov and Ramanujan generalized these results by omitting nuclearity condition [1].

Let  $X = \lambda^\ell(A)$  and  $Y = \lambda^\ell(B)$  be the  $\ell$ -Köthe spaces. Here, we modify Proposition 1 in [1] for  $\ell$ -Köthe spaces and using it we obtain a necessary and sufficient condition in terms of corresponding Köthe matrices when  $(X, Y) \in \mathcal{B}$ . Also we observe a common basic subspace between  $\ell$ -Köthe spaces  $X$  and  $Y$  when  $(X, Y) \notin \mathcal{B}$  and  $(Y, X) \in \mathcal{S}$  following the same lines in [1].

## 2. BOUNDED AND UNBOUNDED OPERATORS IN $\ell$ -KÖTHE SPACES

Let  $\lambda^\ell(A), \lambda^\ell(B)$  be  $\ell$ -Köthe spaces. As in [1] we obtain the following.

**Proposition 2.1.** *Let  $\lambda^\ell(A)$  and  $\lambda^\ell(B)$  be  $\ell$ -Köthe spaces. If there exists a continuous linear unbounded operator  $T : \lambda^\ell(A) \longrightarrow \lambda^\ell(B)$ , then there exists a continuous unbounded quasi-diagonal operator on  $\lambda^\ell(A)$  to  $\lambda^\ell(B)$ .*

*Proof.* Let  $T : \lambda^\ell(A) \longrightarrow \lambda^\ell(B)$  be continuous and unbounded. We may assume without loss of generality that

$$\|Tx\|_k \leq \frac{1}{2^k} \|x\|_k, \quad \forall x \in \lambda^\ell(A)$$

$$\sup_n \frac{\|Te_n\|_{k+1}}{\|e_n\|_k} = \infty, \quad k \in \mathbb{N}.$$

Indeed, one may obtain these by using appropriate multipliers and passing to a subsequence of seminorms, if necessary. Let  $(k_j)$  be a sequence of integers such that each  $k \in \mathbb{N}$  appears in it infinitely many times and choose an increasing subsequence  $(n_j)$  such that

$$\frac{\|Te_{n_j}\|_{k_j+1}}{\|e_{n_j}\|_{k_j}} \geq 2^j, \quad \forall j$$

Let us remind that  $\|\tilde{e}_v\|_k = b_v^k$  and  $\|e_n\|_k = a_n^k$  and let  $Te_n = \sum_v \theta_{nv} \tilde{e}_v$ .

Note that,

$$\begin{aligned} \sup_{|\alpha_v| \leq 1} \left| \sum_v \theta_{nv} \alpha_v \left( \sup_k \frac{b_v^k}{a_n^k} \right) \tilde{e}_v \right| &\leq \sum_k \left( \frac{b_v^k}{a_n^k} \right) \left( \sup_{|\alpha_v| \leq 1} \left| \sum_v \theta_{nv} \alpha_v \tilde{e}_v \right| \right) \\ &\leq \sum_k \frac{1}{a_n^k} \sup_{|\alpha_v| \leq 1} \left| \sum_v \theta_{nv} \alpha_v b_v^k \tilde{e}_v \right| \\ &\leq \sum_k \frac{\|Te_n\|_k}{\|e_n\|_k} \leq \sum_k \frac{1}{2^k} \leq 1 \end{aligned}$$

Therefore we obtain that

$$(2.1) \quad \sup_{|\alpha_v| \leq 1} \left| \sum_v \theta_{n_j v} \alpha_v \left( \sup_k \frac{b_v^k}{a_{n_j}^k} \right) \tilde{e}_v \right| \leq 1 \leq \frac{1}{2^j} \sup_{|\alpha_v| \leq 1} \left| \sum_v \theta_{n_j v} \alpha_v \frac{b_v^{k_j+1}}{a_{n_j}^{k_j}} \tilde{e}_v \right|$$

So there is a  $v_j$  such that

$$t_j := \sup_k \frac{b_{v_j}^k}{a_{n_j}^k} \leq \frac{1}{2^j} \frac{b_{v_j}^{k_j+1}}{a_{n_j}^{k_j}}$$

Otherwise we obtain a contradiction to (2.1) by monotonicity of  $\|\cdot\|$ .

Now, consider the quasi-diagonal operator  $D : \lambda^\ell(A) \longrightarrow \lambda^\ell(B)$  defined by

$$De_{n_j} = t_j^{-1} \tilde{e}_{v_j}, \quad j \in \mathbb{N}$$

$$De_n = 0 \quad \text{if } n \neq n_j$$

Let  $x = \sum_j x_{n_j} e_{n_j} \in \lambda^\ell(A)$ . So,  $Dx = \sum_j x_{n_j} t_j^{-1} \tilde{e}_{v_j}$ . Since  $\left| x_{n_j} t_j^{-1} b_{v_j}^k \right| \leq \left| x_{n_j} a_{n_j}^k \right|$ , by monotonicity we obtain that  $\left\| \left( x_{n_j} t_j^{-1} b_{v_j}^k \right) \right\| \leq \left\| \left( x_{n_j} a_{n_j}^k \right) \right\|$ , i.e.,

$$\|Dx\|_k \leq \|x\|_k \quad \forall k$$

Hence,  $D$  is continuous.

Similarly, it is easy to see that  $D$  is unbounded since for a fixed  $k$ , there is a subsequence  $(j_m)$  such that  $k_{j_m} = k$ ,  $m \in \mathbb{N}$  and

$$\frac{\|De_{n_{j_m}}\|_{k+1}}{\|e_{n_{j_m}}\|_k} \geq 2^{j_m} \rightarrow \infty$$

as  $m \rightarrow \infty$ . This completes the proof.  $\square$

Proposition 2.1 enables us to prove the sufficiency part of the following theorem. Notice that sufficiency can not be obtained directly for a general linear map.

**Theorem 2.2.** *Let  $\lambda^\ell(A)$  and  $\lambda^\ell(B)$  be  $\ell$ -Köthe spaces.  $(\lambda^\ell(A), \lambda^\ell(B)) \in \mathcal{B}$  if and only if for every sequence  $N(k) \uparrow \infty$  there exists  $N \in \mathbb{N}$  such that for each  $r \in \mathbb{N}$  we have  $k_o \in \mathbb{N}$  and  $C > 0$  with*

$$\frac{b_v^r}{a_i^N} \leq C \max_{1 \leq k \leq k_o} \frac{b_v^k}{a_i^{N(k)}}$$

for all  $v \in \mathbb{N}, i \in \mathbb{N}$ .

*Proof.* Suppose  $(\lambda^\ell(A), \lambda^\ell(B)) \in \mathcal{B}$ . Consider  $T : \lambda^\ell(A) \rightarrow \lambda^\ell(B)$  with  $T = e_i' \otimes e_v$  where  $e_i'(x) = x_i$  for all  $x \in \lambda^\ell(A)$ .

Since  $T$  is the operator of rank one, we note that

$$\|T\|_{k, N(k)} = \|e_i'\|_{N(k)} \|e_v\|_k = \frac{b_v^k}{a_i^{N(k)}}$$

Similarly  $\|T\|_{r, N} = \frac{b_v^r}{a_i^N}$ . The result follows from (1.1).

Conversely we want to show that every continuous linear quasi-diagonal operator is bounded. Let  $T : \lambda^\ell(A) \rightarrow \lambda^\ell(B)$  be a continuous quasi-diagonal operator defined by  $T(e_i) = t_i \tilde{e}_{z(i)}$ . By continuity,  $\exists N(k)$  such that

$$\sup_i \frac{\|Te_i\|_k}{\|e_i\|_{N(k)}} = \sup_i \frac{|t_i| b_{z(i)}^k}{a_i^{N(k)}} = C(k) < \infty.$$

Thus for this  $N(k)$ ,  $\exists N \in \mathbb{N}$  such that  $\forall r \in \mathbb{N}$  we have  $k_o \in \mathbb{N}$  and  $C > 0$  with

$$\frac{|t_i| b_{z(i)}^r}{a_i^N} \leq C \max_{1 \leq k \leq k_o} \frac{|t_i| b_{z(i)}^k}{a_i^{N(k)}} \leq C \max_{1 \leq k \leq k_o} C(k).$$

Hence  $\|T\|_{r, N} < \infty$ , i.e.,  $T$  is bounded. In view of Proposition 2.1, we obtain the result.  $\square$

$\lambda^\ell(A)$  and  $\lambda^\ell(B)$  have a common basic subspace if there is a quasi-diagonal operator  $T : X \rightarrow Y$  such that the restriction of  $T$  to some infinite dimensional basic subspace of  $X$  is an isomorphism. We observe the following extension of Proposition 3 in [1] to the  $\ell$ -Köthe space case. The proof is the same as in [1].

**Corollary 2.3.** *If  $(\lambda^\ell(B), \lambda^\ell(A)) \in \mathcal{S}$  and there exists a continuous unbounded operator  $T : \lambda^\ell(A) \rightarrow \lambda^\ell(B)$ , then  $\lambda^\ell(A)$  and  $\lambda^\ell(B)$  have a common basic subspace.*

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